Separation of Variables for $\frac{dy}{dt} = f(t)g(y)$

Step 1. Separate the variables: divide through by $g(y)$

$$\frac{1}{g(y)}\frac{dy}{dt} = f(t)$$ provided $g(y) \neq 0$ (we shall talk about this later)

Step 2. Integrate both sides with respect to time

$$\int \frac{1}{g(y)}\frac{dy}{dt}\,dt = \int f(t)\,dt$$

Step 3. Cancel "*dt*"s on left side

$$\boxed{\int \frac{1}{g(y)}\,dy = \int f(t)\,dt}$$

Separation of variables formula

Step 4. Do the integrations (don't forget the constant of integration)

Step 5. If possible, rearrange to give *y* in terms of *t*

turns out that we only need one constant of integration, e.g. write +*C* on right hand side

Example: Newton's Law of Cooling

Rate of change of temperature of object is proportional to the difference between the ambient temperature (*T*) and the temperature of the object (*y*). Initial temperature of object is y_0. Write down and solve an initial value problem describing this situation. Interpret your solution. What happens to the temperature of the object in the long run?

Solve $\frac{dy}{dt} = k(T-y)$; $y(0) = y_0$

$$\frac{1}{T-y}\frac{dy}{dt} = k$$

$$\int \frac{1}{T-y}\frac{dy}{dt}\,dt = \int k\,dt$$

$$\int \frac{1}{T-y}\,dy = \int k\,dt$$

$$-\ln|T-y| = kt + C$$

$$\ln|T-y| = -kt - C$$

$$e^{(\ )} \qquad |T-y| = e^{-kt-C}$$

$e^{(\ )}$

$$|T-y| = e^{-kt-C}$$

$$T-y = \pm e^{-kt-C}$$

$$T-y = \pm e^{-C} \cdot e^{-kt}$$

write $A = \pm e^{-C}$

$$T-y = Ae^{-kt}$$

$$y-T = -Ae^{-kt}$$

$$y = T - Ae^{-kt}$$

$$y(t) = T - Ae^{-kt}$$

$$y(0) = y_0$$

$$y_0 = T - Ae^{-k \cdot 0}$$

$$y_0 = T - A$$

$$A = T - y_0$$

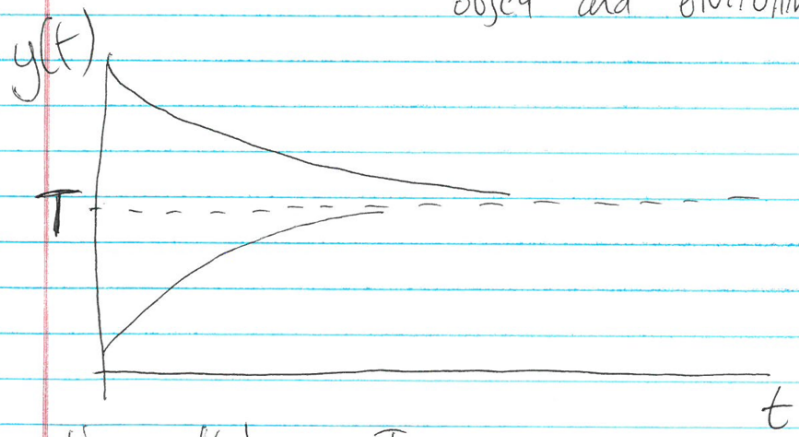
$$y(t) = T - (T-y_0)e^{-kt}$$

$$\boxed{y(t) = T + (y_0 - T)e^{-kt}}$$

exponentially decaying

ambient temp.

initial temperature difference between object and environment.



$$\lim_{t \to \infty} y(t) = T$$