BMA 771 Sample Mid-Term Test

Time allowed: one hour and fifteen minutes.

You may not use the textbook, any notes or a computer while doing the test.

Your answers must contain enough detail for your methods and reasoning to be clear.

Even if you cannot answer one part of a question, you may still be able to answer later parts of the question.

You may answer the questions in any order you choose.

Question ONE

Consider the model

$$dx/dt = rx + x^3 - x^5,$$

where r is a parameter that can take any real value.

(a) Show that the model has 1, 3 or 5 equilibria, depending on the value of r.

(Hint: you can solve a quartic of the form $0 = a + bX^2 + cX^4$ by considering it as a quadratic in X^2.)

(b) Sketch all qualitatively different forms that the vector field can take, paying close attention to borderline cases.

(c) Sketch the bifurcation diagram of the model, labeling each bifurcation point with its location and type.

(d) With reference to your bifurcation diagram, explain what is meant by bistability and hysteresis.

Question TWO

The concentration, $g(t)$, of a certain protein can be described by the model

$$dg/dt = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2},$$

where each k is a positive constant and s_0 is the concentration of a chemical that acts as a signal.

(a) Show that the system can be put in the non-dimensional form

$$dx/d\tau = s - rx + \frac{x^2}{1 + x^2},$$

where r is positive and s is non-negative.

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(b) Show that, when $s = 0$, there are two positive equilibria, provided that $r < r_c$. Find the value of r_c.

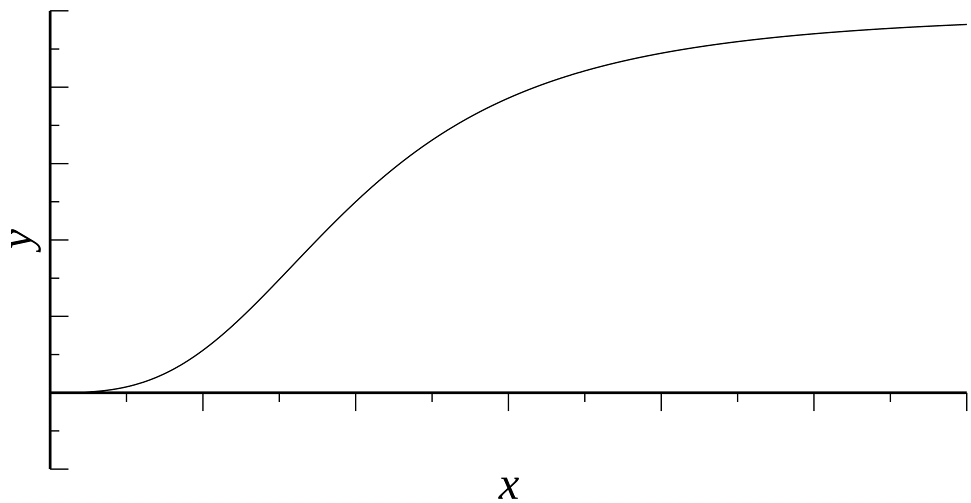
What bifurcation happens when $r = r_c$? Use a sketch that includes the curve $y = x^2/(1 + x^2)$ to show graphically what happens at this bifurcation. (You may use one of the graphs provided below.)

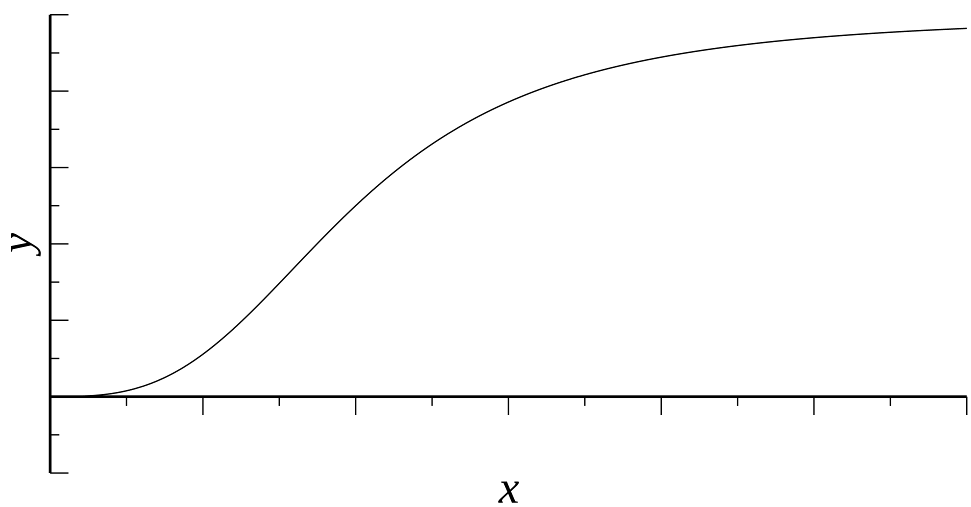
(c) Now suppose that $0 < r < r_c$, and that, initially, $s = 0$ and $g = 0$.

What happens to x as s is slowly increased? Explain, using a sketch that includes the curve $y = x^2/(1+x^2)$, why the value of x will, above a certain value of s (which you need not find), jump to a high value.

What would happen if s were then decreased?

Plots of $y = x^2/(1 + x^2)$ that you may use for parts (b) and (c).





Question THREE

Consider the flow on the circle defined by the differential equation

$$\frac{d\theta}{dt} = \sin\theta(\mu - \sin\theta).$$

Here μ is a parameter that can take any real value. As usual, θ is the phase variable (angle) on the circle.

The qualitative behavior of the system depends on the parameter μ. Illustrate each of the different possible behaviors seen by sketching the vector field (or phase portrait) on the circle for each of these possibilities. Your collection of sketches should include sketches for the parameter values at which bifurcations occur. The stabilities of all equilibrium points should be made clear.

List the bifurcations that this model exhibits and the parameter values at which they occur. Sketch the bifurcation diagram for the system, plotting θ (vertical) against μ (horizontal).

(20)

(You may answer the various parts of this question in any order that you choose, as long as your reasoning is clear.)