4.1.1 $\dot{\theta} = \sin a\theta = f(\theta)$

to uniquely define a vector field on the circle, we need $f(\theta + 2\pi) = f(\theta)$

i.e. $\sin a(\theta + 2\pi) = \sin a\theta$

use angle addition formula

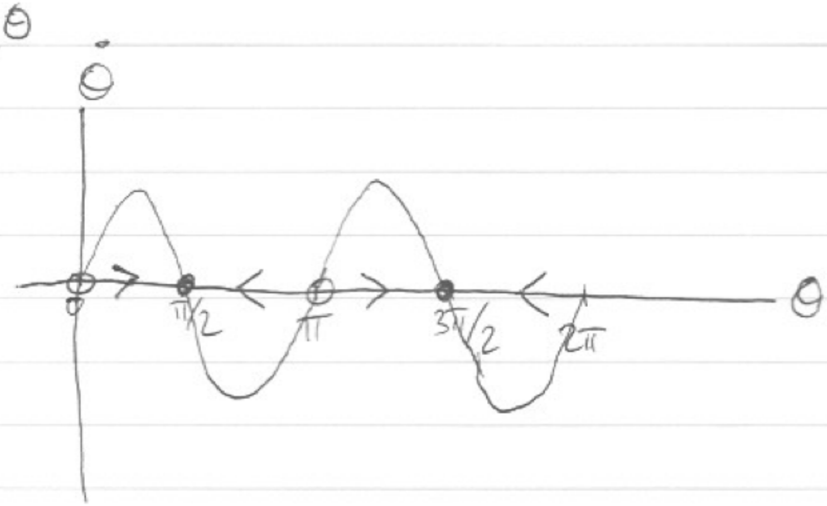
$$\sin a(\theta + 2\pi) = \sin(a\theta + 2\pi a)$$
$$= \sin a\theta \cos 2\pi a + \cos a\theta \sin 2\pi a$$

so require $\cos 2\pi a = 1, \quad \sin 2\pi a = 0$

$\Rightarrow \quad 2\pi a = 2\pi n \qquad n$ is an integer

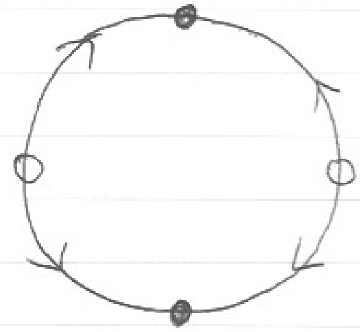
hence, a must be an integer.

4.1.3 $\dot{\theta} = \sin 2\theta$

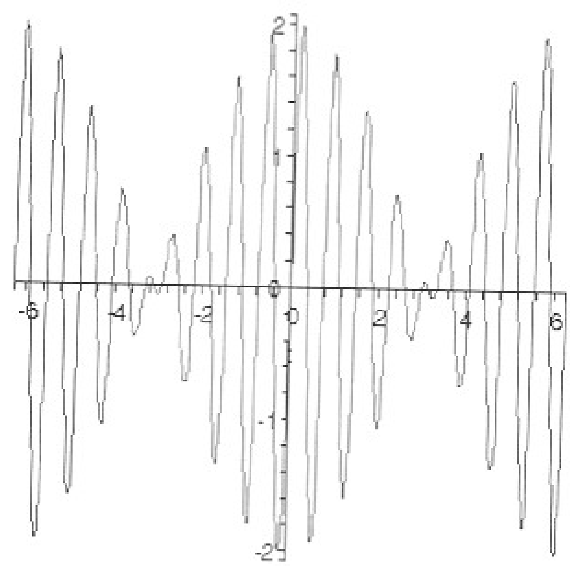


$\theta = 0, \pi$ are unstable fixed points
$\theta = \pi/2, 3\pi/2$ are stable fixed points

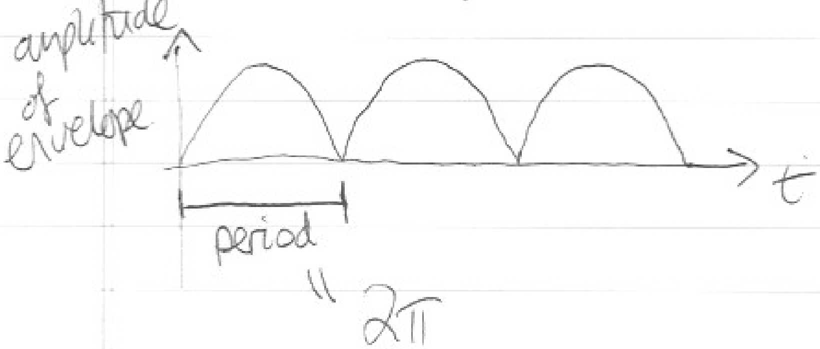
phase portrait on circle



4.2.2



a) $\longrightarrow$ read off the period from the graph.



b) use $\sin\alpha + \sin\beta$

$$= 2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta)$$

$$x(t) = \sin 8t + \sin 9t$$

$$= 2\sin\tfrac{17}{2}t\,\cos\tfrac{1}{2}t \qquad (\text{NB } \cos(-t) = \cos t)$$

rapid oscillations

amplitude modulation.

one period of the amplitude modulation sees $\cos\frac{1}{2}t$ rise from zero to one and back again. $\boxed{\text{see graph in (a)}}$
or goes from 1 to 0 and back again

$\Rightarrow$ this occurs as $\frac{1}{2}t$ increases ~~from zero to~~ from 0 to π

$\Rightarrow t$ goes from 0 to 2π

$\therefore$ period 2π

4.2.3 It's the two joggers example.

Minute hand has period 1 hr, hour hand has period 12 hours. So, from the formula in the notes,

$T = (1/T_1 - 1/T_2)^{-1}$ = 12/11 hours.

I think it would be better to spell out the detail (which isn't much more to do)…

$$\dot{\theta}_1 = \omega_1 = \frac{2\pi}{1} = 2\pi \Rightarrow \theta_1 = 2\pi t$$

$$\dot{\theta}_2 = \omega_2 = \frac{2\pi}{12} = \pi/6 \Rightarrow \theta_2 = \pi t/6$$

$$\theta_1 = \theta_2 + 2\pi \Rightarrow 2\pi t = \frac{\pi t}{6} + 2\pi$$
$$\Rightarrow 11\pi t/6 = 2\pi$$

again giving 12/11 hours.

Intuitive methods

First method: We know that the hands are aligned at noon and again at midnight, a twelve hour period during which they cross 11 times. (At 1:something, 2:something, 3:something, … 10:something, 11:something—except that the 11:something one must be at midnight). So there are 12/11 hours between crossings.

Second method: Imagine the minute hand trying to catch up to the hour hand after the first hour has been completed. We keep track of a sequence of steps in which the minute hand moves to the previous location of the hour hand.

Note: minute hand completes a revolution in 1 hour, hour hand completes a revolution in 1/12 hour, so is rotating at 1/12 revolution per hour.

After 1 hour, minute hand has completed 1 revolution, hour hand has completed 1/12 of a revolution. Minute hand needs to catch up 1/12 revolution.

Minute hand takes 1/12 hour to make 1/12 revolution. But in this time, hour hand has moved $(1/12)^2$ revolutions beyond its previous location.

Minute hand takes $(1/12)^2$ hours to move to previous location of hour hand. But in this time, hour hand has moved $(1/12)^3$ revolutions ahead.

Minute hand takes $(1/12)^3$ hours to move to this location, but the hour hand is then $(1/12)^4$ ahead, and so on…

Notice that the two locations converge as the number of steps goes to infinity.

The total time taken for these steps is

$$1 + \frac{1}{12} + \left(\frac{1}{12}\right)^2 + \left(\frac{1}{12}\right)^3 + \cdots = \frac{1}{1 - \frac{1}{12}} = \frac{12}{11} \text{ hours}$$

(c.f. Zeno's paradox of Achilles and the tortoise) alternatively

$$\dot{\theta}_1 = \omega_1 = 2\pi/1 = 2\pi \qquad \theta_1 = 2\pi t$$
$$\dot{\theta}_2 = \omega_2 = 2\pi/12 = \pi/6 \qquad \theta_2 = \frac{\pi}{6}t$$

$$\theta_1 = \theta_2 + 2\pi \Rightarrow 2\pi t = \frac{\pi}{6}t + 2\pi$$

$$\Rightarrow \frac{11\pi}{6}t = 2\pi$$

$$\Rightarrow t = \frac{12}{11} \text{ hours.}$$

4.3.3 $\dot{\theta} = \mu \sin\theta - \sin 2\theta$

$= \mu \sin\theta - 2\sin\theta\cos\theta$

$= \sin\theta(\mu - 2\cos\theta)$

$\dot{\theta} = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$ or π

or $\mu = 2\cos\theta$

this only has solutions if $-2 \leq \mu \leq 2$

2 equilibria if $|\mu| \geq 2$

4 equilibria if $|\mu| < 2$

~~(when $\mu = \pm 2$, the solutions of $\mu = 2\cos\theta$ are also $\theta = 0$ and π)~~

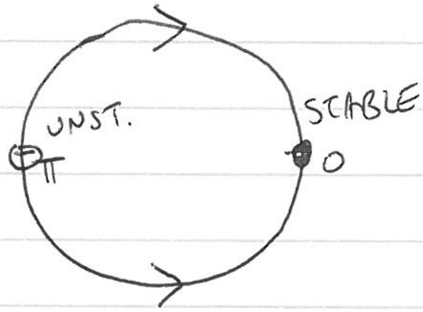
when $\mu = 2$, the solution of $\mu = 2\cos\theta$ is $\theta = 0$

when $\mu = -2$, the solution is $\theta = \pi$

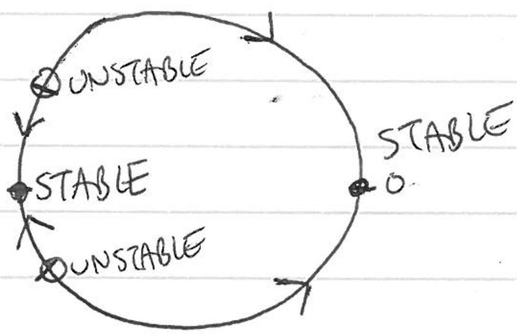
$\mu < -2$ $\quad (\mu - 2\cos\theta) < 0$ for all θ, so sign of $\dot{\theta}$ (directions of arrows) depends on sign of $-\sin\theta$

$\downarrow$

< 0 for θ between 0 and π, > 0 for θ between π and 2π



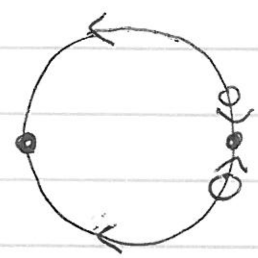
$-2 < \mu < -2$



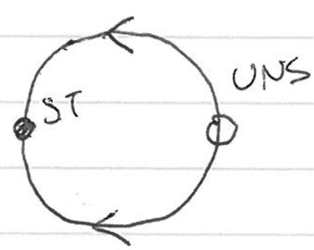
notice: when μ is close to -2, both solutions of $\mu = 2\cos\theta$ are close to $\theta = \pi$

when $\mu = 0$, the solutions of $\mu = 2\cos\theta$ are $\theta = \pi/2, 3\pi/2$

as μ approaches 2 we have

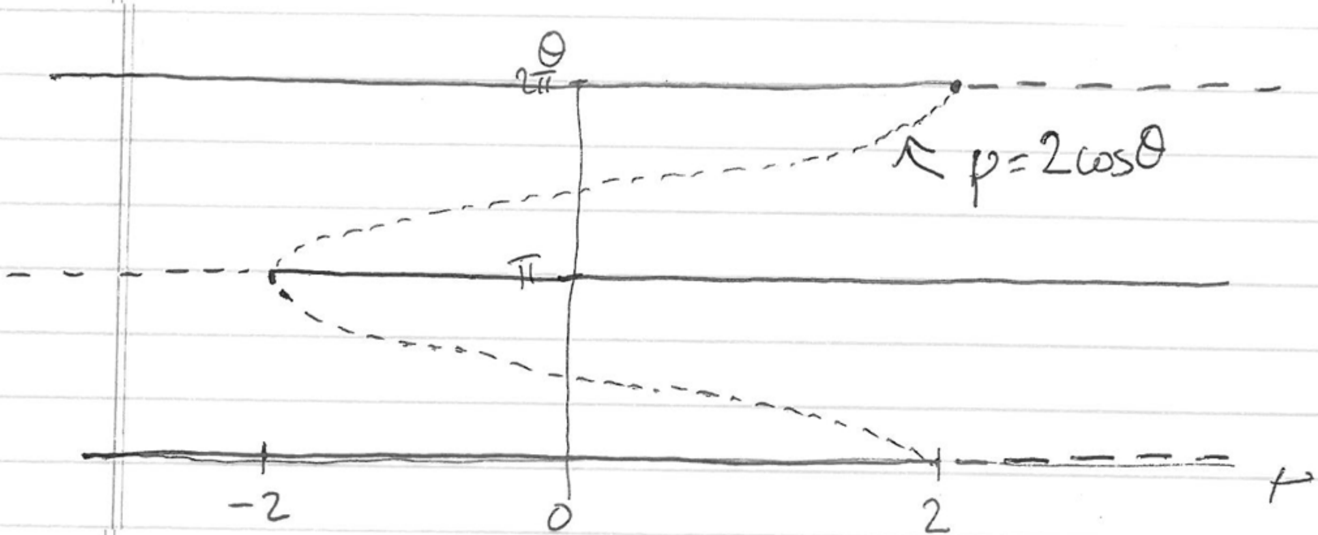


$\mu > 2$



These are pitchfork bifurcations at $\mu=+2, \mu=-2$.

Both are subcritical.



The pitchfork at $\mu=2$ is easier to see when we remember that 0 and 2π are the same point - we could wrap up the θ direction onto a circle.