Separable Systems

A separable ODE is of the form:

$$\frac{dx}{dt} = f(x)g(t).$$

(Notice that $dx/dt = f(x)$ and $dx/dt = g(t)$ are special cases of separable ODEs.)

Provided $f(x) \neq 0$, can divide through by $f(x)$:

$$\frac{1}{f(x)}\frac{dx}{dt} = g(t)$$

$$\Rightarrow \int \frac{1}{f(x)}\frac{dx}{dt}\,dt = \int g(t)\,dt \qquad \text{(integrate over time)}$$

$$\Rightarrow \boxed{\int \frac{1}{f(x)}\,dx = \int g(t)\,dt} \qquad \text{(last step is integration by substitution/use of chain rule)}$$

IF we can integrate $1/f(x)$ and $g(t)$, then we will find a relationship between x and t.

Important: An **equilibrium** occurs if there is an x^* for which $f(x^*) = 0$, because then dx/dt will equal zero and so the value of x will not change over time.

Simplest Example: Exponential Growth/Decay

$$\frac{dN}{dt} = rN \qquad \text{rate of change (e.g. of population size) is proportional to population size}$$

This is a separable equation, e.g. with $f(N) = N$ and $g(t) = r$.

$$\Rightarrow \frac{1}{N}\frac{dN}{dt} = r$$

$$\Rightarrow \int \frac{1}{N}\frac{dN}{dt}\,dt = \int r\,dt$$

$$\Rightarrow \int \frac{1}{N}\,dN = rt + c \qquad c \text{ is constant of integration}$$

$$\Rightarrow \ln|N| = rt + c$$

$$\Rightarrow N = e^{rt+c}$$

$$= e^{rt}e^{c}$$

$$= Ae^{rt}. \qquad \text{Writing } e^c \text{ as } A.$$

When $t = 0$, $N = A$, so $N(t) = N(0)e^{rt}$, where $N(0)$ is the number at time $t = 0$.

Exponential growth if $r > 0$, exponential decay if $r < 0$.

Geometric view of this equation

Example: Drug Metabolism

A doctor administers some drug to a patient by injecting at a constant rate r ($r > 0$).

The body metabolizes the drug, at a rate proportional to the concentration of the drug ("first order reaction"). Write the constant of proportionality as k ("rate constant", $k > 0$).

Let $X(t)$ stand for the concentration of drug at time t.

For notational convenience, let's write $R = r/k$, so that $r = Rk$.

$$\frac{dX}{dt} = r - kX \qquad \text{note the signs of the terms}$$

$$\Rightarrow \quad \frac{dX}{dt} = Rk - kX = -k(X - R).$$

This is a linear, inhomogeneous ODE (if $R = 0$, it would be homogeneous). It is also separable, with $f(X) = X - R$ and $g(t) = -k$.

$$\Rightarrow \quad \frac{1}{X - R}\frac{dX}{dt} = -k$$

$$\Rightarrow \quad \int \frac{1}{X - R}\, dX = \int -k\, dt$$

$$\Rightarrow \quad \ln|X - R| = -kt + c$$

$$\Rightarrow \quad |X - R| = e^{-kt+c}$$

$$\Rightarrow \quad X - R = \pm e^{c} e^{-kt}$$

$$= Ae^{-kt} \qquad \text{Writing } A = \pm e^{c}$$

$$\Rightarrow \quad X = R + Ae^{-kt}.$$

Setting $t = 0$, we see that the value of A is equal to $X(0) - R$.

$$X(t) = \frac{r}{k} + \left(X(0) - \frac{r}{k}\right) e^{-kt}.$$

The difference between the concentration of the drug and r/k decays exponentially over time.

The drug concentration approaches the value r/k in the long term, either from above or below, depending on whether the initial concentration of the drug is above or below r/k.

Geometric view of this equation

(In the Strogatz book, this example is presented as "Newton's Law of Heating and Cooling", but the math is identical.)

Another Separable Model: Age-Dependent Death Process

Imagine following a group of N_0 individuals born at time $t = 0$, and that they experience an age-dependent (per-capita) death rate equal to $\mu(a)$, where a is their age.

How many individuals remain alive at time T?

Because they were born at time $t = 0$, their age equals the time, so $a = t$, and we have

$$\frac{dN}{dt} = -\mu(t)N.$$

This is separable, so

$$\begin{aligned}
& \int_{t=0}^{t=T} \frac{1}{N}\frac{dN}{dt}\,dt = \int_0^T -\mu(t)\,dt \\
\Rightarrow\quad & \int_{N_0}^{N(T)} \frac{1}{N}\,dN = -\int_0^T \mu(t)\,dt \\
\Rightarrow\quad & \ln N(T) - \ln N_0 = -\int_0^T \mu(t)\,dt \\
\Rightarrow\quad & N(T) = N_0 e^{-\int_0^T \mu(t)\,dt}
\end{aligned}$$

The exponential term represents the probability that a given individual will survive to age T (or greater)— the "survivorship function".

For example, take a constant death rate $\mu(t) \equiv \mu$: this gives $N(T) = N_0 e^{-\mu T}$.