Ordinary Differential Equation (ODE) models: the focus of this course

One independent variable, usually time:

- $\frac{dx}{dt} = f(x,t)$ the rate of change of x is a function of x and time

  one state variable: one dimensional system

- $\frac{dx}{dt} = f(x,y,t)$, $\frac{dy}{dt} = g(x,y,t)$ coupled ODEs, two state variables: two dimensional system

> **Partial differential equations** involve two or more independent variables (such as time and space), e.g.
> $$\frac{\partial y}{\partial t} = f(y,t,x) + D\frac{\partial^2 y}{\partial x^2}$$
> where x represents a spatial co-ordinate. The second term on the right hand side of this **reaction-diffusion equation** depicts diffusion of y in space. PDEs are the topic of BMA 774

A **first order ODE** only involves first derivatives. (Remember the alternative notation for time derivatives: $dx/dt = \dot{x}$)

Second and **higher order ODEs** involve higher order derivatives.

Second order ODEs commonly arise in physics because acceleration is a second derivative. Classic examples:

- $\frac{d^2x}{dt^2} = -\omega^2 x$ describes the motion of a mass on a (perfect) spring: the simple harmonic oscillator.

- $\frac{d^2x}{dt^2} = -\frac{g}{L}\sin x$ describes the simple pendulum (x is the angle to the vertical)

  (Since $\sin x \approx x$ when x is small, the simple harmonic oscillator provides a good approximation to the pendulum model when the oscillations have small amplitude.)

We focus on Systems of First Order Ordinary Differential Equations (coupled 1st order ODEs)

- These provide a natural description of many biological systems

- **We can always reduce an ODE of any order to a set of first order ODEs**.

  Example: $\frac{d^2x}{dt^2} = -\omega^2 x$

We use a trick: write $y = \frac{dx}{dt}$, because this gives $\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dy}{dt}$. This lets us rewrite the 2nd order derivative as a first order derivative, and hence our 2nd order ODE as a coupled pair of 1st order ODEs (i.e. a two dimensional system):

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -\omega^2 x$$

This approach can be extended to reduce an nth order ODE to n coupled 1st order ODEs.

General form of n dimensional first order model

$$\frac{dx_1}{dt} = f_1\,(x_1, x_2, \ldots, x_n, t)$$
$$\frac{dx_2}{dt} = f_2\,(x_1, x_2, \ldots, x_n, t)$$
$$\vdots$$
$$\frac{dx_n}{dt} = f_n\,(x_1, x_2, \ldots, x_n, t)$$

Model is **linear** if the RHS are linear combinations of the x_i , i.e. are of the form $f_i\,(x_1, x_2, \ldots, x_n, t) = \alpha_{i1}x_1 + \ldots + \alpha_{in}x_n + \beta_i$.

This means the x_i only appear raised to the first power and there are no products of two or more x_i.
NOTE: The α and β can be functions of time **but not of *x***.

Linear: $$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -\omega^2 x\end{aligned}$$ Simple harmonic oscillator

Nonlinear: $$\frac{dx}{dt} = -x^2 \qquad \begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -xy\end{aligned} \qquad \begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -\frac{g}{L}\sin x\end{aligned}$$ Simple pendulum

Linear systems are easier to solve than nonlinear systems. (Many math courses will focus on linear ODEs for this reason.)

Reason: in some sense, we can break down a linear problem into smaller parts, each of which can be solved more easily, and can then put the parts back together to give a solution to the whole problem. This is not the case for nonlinear systems.

(Replacing $\sin x$ by x in the nonlinear pendulum model gives the linear SHO model. "Linearization" takes us from a model that is not easy to solve to a model that is easy to solve. This is an important approach that we will use quite often in 771.)

Autonomous and Non-Autonomous Systems

If the right hand side of the ODE does not depend **explicitly** on time, *i.e.* t does not appear on the RHS, then the system is **autonomous**. Otherwise, it is **non-autonomous**.

Some people use the term **forced** to describe a non-autonomous system. One common form of forcing involves a periodic function, e.g. a forced pendulum. In some contexts we talk about "seasonal forcing".

(The right hand side of an autonomous ODE depends **implicitly** on time via the state variables.)

We will focus on autonomous systems. (It turns out that they have some nice properties…)

We can always convert a non-autonomous system to an autonomous one, at the cost of adding an extra dimension.

For example, $\dot{x} = f(x,t)$.

Write $\dot{z} = 1$, which gives $z = t + c$. c is some constant; set it equal to zero.

Then we can rewrite the non-autonomous ODE as the autonomous system: $$\begin{aligned}\dot{x} &= f(x,z) \\ \dot{z} &= 1\end{aligned}$$